Hyperbolic formulations and numerical relativity II: Asymptotically constrained system of the Einstein equation *

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We study asymptotically constrained systems for numerical integration of the Einstein equation, which is intended to be robust against perturbative errors for the free evolution of the initial data. We, first, examine the previously proposed "λ-system", which introduces artificial flows to constrained surfaces based on the symmetric hyperbolic formulation. We show that this system works as expected for the wave propagation problem in the Maxwell system and in general relativity using Ashtekar's connection formulation. We, second, propose a new mechanism to control the stability, which we named "adjusted system". This is simply obtained by adding constraint terms in the dynamical equations and adjusting its multipliers. We explain why a particular choice of multiplier reduces the numerical errors by non-positiveness (or non-zero) of the eigenvalues of the adjusted constraint propagation equations. This "adjusted system" is also tested in the Maxwell system and in the Ashtekar's system. This mechanism affects more than the system's symmetric hyperbolicity.

I. INTRODUCTION

Numerical relativity, an approach to solve the Einstein equation numerically, is supposed to be the only way to study the highly non-linear gravitational phenomena. However, we still do not have the definite recipe for integrating Einstein equation, which enables us the accurate and long-time stable time evolutions. Here and hereafter, we mean the stable evolution by the statement that the system keeps the violation of the constraint within a suitable small value in its free numerical evolution.

As the authors discussed in our preceding paper (Paper I) [1], one direction for obtaining more stabler system is to apply a set of dynamical equations which reveal hyperbolic form (or first-order form). The standard Arnowitt-Deser-Misner (ADM) formulation does not have this feature, but there are many alternative proposals for constructing hyperbolic set of equations (references are in [1]). We, however, showed in the Paper I that a symmetric hyperbolic form (which is the ultimate level of hyperbolicity) does not necessary show the best performance for stable numerical evolution compared with weakly and strongly hyperbolic systems. This experiment was performed using the Ashtekar's connection formulation of general relativity, because one can keep the same fundamental dynamical variables when one compares the three different levels of hyperbolic formulations.

In this article, we discuss another (but somewhat related) approaches to obtain the stable evolution of the Einstein equation. The idea is to construct a robust system against a perturbative error which is produced during numerical time integration. The contents are divided in two separate ones.

The first one is so-called "λ-system", which was proposed originally by Brodbeck, Frittelli, Hübner and Reula (BFHR) [2]. The idea of this approach is to introduce additional variables, λ, which indicates the violation of the constraints, and to construct a symmetric hyperbolic system for both the original variables and λs together with imposing dissipative dynamical equations for λs. BFHR constructed their λ-system based on Frittelli-Reula's symmetric hyperbolic formulation of the Einstein equation [3], and we [4] have also presented the similar system for the Ashtekar's connection formulation [5] based on its symmetric hyperbolic expression [6,7]. In §II, we review this system and present numerical examples which show this system behaves as expected.

The second one comes from the same motivation but turns to be more practical, which we propose as "adjusted-system". The essential procedures are to to add constraint terms in the right-hand-side of the dynamical equations with multipliers, and to choose the multipliers so as they decrease the violation of constraint equation. This second step will be explained by obtaining non-positive (or non-zero) eigenvalues of the adjusted constraint propagation equations. We remark that adjusting the dynamical equation using the constraints is not a new idea. This can be

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seen for example in a remedial ADM system by Detweiler [8] or a conformally decoupled trace-free re-formulation of ADM by Nakamura *et al* [9] (the advantages of the latter system were also reported in [10,11]). We also remark that this eigenvalue criterion is also the core part of the theoretical support of the above λ-system. In §III, we describe this approach and present numerical examples again in the Maxwell system and in the Ashtekar's system.

This "adjusted-system" does not change the number of dynamical variables, and does not require hyperbolicity in the original set of equation. Therefore we think our results promote further applications in numerical relativity.

We do not repeat our explanations on Ashtekar's connection formulation in our notation, nor our detail numerical procedures in this article, since they are described in our Paper I [1].

II. ASYMPTOTICALLY CONSTRAINED SYSTEM 1: λ-SYSTEM

We begin by reviewing the fundamental procedures of the "λ-system" proposed by Brodbeck, Frittelli, Hübner and Reula (BFHR) [2]. We, then, demonstrate how this system works in Maxwell equations, and Ashtekar's connection formulation of the Einstein equations in the following subsections.

A. "λ system"

The actual procedures for constructing a λ system are followings.

(1) Prepare a symmetric hyperbolic evolution system which describe the problem; say

$$\partial_t u^\gamma = A^{i\gamma}{}_\delta \partial_i u^\delta + B^\gamma, \tag{2.1}$$

where u^γ ($\gamma = 1, \ldots, N$) is a set of dynamical variables, $A(u(x^i))$ forms a symmetric matrix (Hermitian matrix when u is complex variables) and $B(u(x^i))$ is a vector, where A and B do not include any further spatial derivatives in these components. The system may have constraint equations, which should be the first class. Ideally, we expect that the evolution equation of the set of constraints C^ρ ($\rho = 1, \ldots, M$), which hereafter we denote constraint propagation equation, forms a first order hyperbolic formulation (cf. [12]), say

$$\partial_t C^\rho = D^{i\rho}{}_\sigma \partial_i C^\rho + E^\rho{}_\sigma C^\sigma, \tag{2.2}$$

(where D, E are the same with A, B above) but this hyperbolicity may not be necessary.

(2) Introduce λ^ρ as an indicator of violations of the constraint equations, $C^\rho \approx 0$. ($\approx$ denotes weakly equal.) We impose that λ obeys a dissipative equations of motion

$$\partial_t \lambda^\rho = \alpha_{(\rho)} C^\rho - \beta_{(\rho)} \lambda^\rho \text{ (we do not take sum about } \rho \text{ and } (\rho) \text{ in right hand side)} \tag{2.3}$$

with the initial data $\lambda^\rho = 0$, and by setting $\alpha \neq 0, \beta > 0$. We remark that λ^ρ keeps as zero during the time evolutions if there is no violations of the constraints.

(3) Take a set of (u, λ) as a dynamical variables, and modify these evolution equations so as to form a symmetric hyperbolic system. That is, the set of equation

$$\partial_t \begin{pmatrix} u^\gamma \\ \lambda^\rho \end{pmatrix} \cong \begin{pmatrix} A^{i\gamma}{}_\delta & 0 \\ F^{i\rho}{}_\delta & 0 \end{pmatrix} \partial_i \begin{pmatrix} u^\delta \\ \lambda^\sigma \end{pmatrix}, \tag{2.4}$$

($\cong$ means that we have extracted only the term which appears in the principal part of the system) can be modified as

$$\partial_t \begin{pmatrix} u^\gamma \\ \lambda^\rho \end{pmatrix} \cong \begin{pmatrix} A^{i\gamma}{}_\delta & \bar{F}^i{}_\sigma{}^\gamma \\ F^{i\rho}{}_\delta & 0 \end{pmatrix} \partial_i \begin{pmatrix} u^\delta \\ \lambda^\sigma \end{pmatrix}, \tag{2.5}$$

of which additional terms will not disturb the hyperbolicity of equations of u^γ, rather they make the whole system symmetric hyperbolic, which guarantees the well-posedness of the system.

Therefore the obtained system, (2.5), is supposed to have unique solution. If there occurs a perturbative violation of constraints, $\lambda^\rho \neq 0$, during its evolution, by choosing an appropriate αs and βs in (2.3), λs are expected to decay to zero, which means the total system evolves into the constrained surface asymptotically. We note that this procedure requires that the original system u forms a symmetric hyperbolic system, so that applications to the Einstein equation are somewhat restricted. BFHR [2] constructed this λ-system using a Frittelli-Reula's formulation [3]. We [4] also applied this system for the symmetric hyperbolic version of Ashtekar's formulation [6].

We next review a proof briefly why the system (2.5) ensures the evolution to be constrained asymptotically. We first remark again that we only consider perturbative violations of constraints in our evolving system. Steps are followings.

(a) Since we modify the equations for u^γ, the propagation equation of the constraints are also modified; write them schematically as

$$\partial_t C^\rho = D^{i\rho}{}_\sigma \partial_i C^\sigma + E^\rho{}_\sigma C^\sigma + G^{ij\rho}{}_\sigma \partial_i \partial_j \lambda^\sigma + H^{i\rho}{}_\sigma \partial_i \lambda^\sigma + I^\rho{}_\sigma \lambda^\sigma. \tag{2.6}$$

(b) In order to see the asymptotic behaviors of (λ^ρ, C^ρ), we write them using their Fourier components so that their evolution equations become homogenous form. That is, we transform (λ^ρ, C^ρ) to $(\hat\lambda^\rho, \hat C^\rho)$ as

$$\lambda(x,t)^\rho = \int \hat\lambda(k,t)^\rho \exp(ik\cdot x) d^3k, \quad C(x,t)^\rho = \int \hat C(k,t)^\rho \exp(ik\cdot x) d^3k, \tag{2.7}$$

then we see the evolution equations (2.3) and (2.6) becomes

$$\partial_t \begin{pmatrix} \hat\lambda^\rho \\ \hat C^\rho \end{pmatrix} = \begin{pmatrix} -\beta_{(\rho)}\delta^\rho_\sigma & \alpha_{(\rho)}\delta^\rho_\sigma \\ -G^{ij\rho}{}_\sigma k_i k_j + iH^{i\rho}{}_\sigma k_i + I^\rho{}_\sigma & iD^{i\rho}{}_\sigma k_i + E^\rho{}_\sigma \end{pmatrix} \begin{pmatrix} \hat\lambda^\sigma \\ \hat C^\sigma \end{pmatrix} := P \begin{pmatrix} \hat\lambda^\sigma \\ \hat C^\sigma \end{pmatrix}. \tag{2.8}$$

(c) If all eigenvalues of this coefficient matrix P have negative real part, a pair $(\hat\lambda, \hat C)$ evolves as $\exp(-\Lambda t)$ asymptotically where $-\Lambda$ is diagonalized matrix of P, which indicates that the original variables (λ, C) evolves similarly. It would be the best if we could determine αs and βs in such a way in general, but it is not possible. Therefore we extract the principal order of P and examine the condition for αs and βs so as P only has negative (real) eigenvalues. We remark again that this procedure is justified when we only consider a perturbative error from the constraint surface.

B. Example 1: Maxwell equations

As the first example, we demonstrate the Maxwell equation with λ-system. The Maxwell equation forms a linear and symmetric hyperbolic dynamical equation, together with two constraint equation. This might be the best system to start with.

1. *λ-system*

Maxwell equations for electric field E^i and magnetic filed B^i in the vacuum space consists from two constraint equations,

$$C_E := \partial_i E^i \approx 0, \tag{2.9}$$
$$C_B := \partial_i B^i \approx 0, \tag{2.10}$$

and a dynamical equations,

$$\partial_t \begin{pmatrix} E^i \\ B^i \end{pmatrix} = \begin{pmatrix} 0 & -c\epsilon^i{}_j{}^l \\ c\epsilon^i{}_j{}^l & 0 \end{pmatrix} \partial_l \begin{pmatrix} E^j \\ B^j \end{pmatrix}, \tag{2.11}$$

which satisfies a symmetric hyperbolicity. Constraint evolutions become $\partial_t C_E = 0$ and $\partial_t C_B = 0$, which means (trivial) symmetric hyperbolicity. According to the procedures, we introduce λs which obey

$$\partial_t \lambda_E = \alpha_1 C_E - \beta_1 \lambda_E, \tag{2.12}$$
$$\partial_t \lambda_B = \alpha_2 C_B - \beta_2 \lambda_B, \tag{2.13}$$

with the initial data $\lambda_E = \lambda_B = 0$ and take $(E, B, \lambda_E, \lambda_B)$ as a set of variables to evolve:

$$\partial_t \begin{pmatrix} E^i \\ B^i \\ \lambda_E \\ \lambda_B \end{pmatrix} = \begin{pmatrix} 0 & -c\epsilon^i{}_j{}^l & 0 & 0 \\ c\epsilon^i{}_j{}^l & 0 & 0 & 0 \\ \alpha_1 \delta^l{}_j & 0 & 0 & 0 \\ 0 & \alpha_2 \delta^l{}_j & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} E^j \\ B^j \\ \lambda_E \\ \lambda_B \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\beta_1 \lambda_E \\ -\beta_2 \lambda_B \end{pmatrix}. \tag{2.14}$$

We obtain immediately an expected symmetric form as

$$\partial_t \begin{pmatrix} E_i \\ B_i \\ \lambda_E \\ \lambda_B \end{pmatrix} = \begin{pmatrix} 0 & -c\epsilon^{ijl} & \alpha_1 \delta^{li} & 0 \\ c\epsilon^{ijl} & 0 & 0 & \alpha_2 \delta^{li} \\ \alpha_1 \delta^{lj} & 0 & 0 & 0 \\ 0 & \alpha_2 \delta^{lj} & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} E_j \\ B_j \\ \lambda_E \\ \lambda_B \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\beta_1 \lambda_E \\ -\beta_2 \lambda_B \end{pmatrix}. \tag{2.15}$$

The last step is to determine α_1, α_2, β_1, and β_2

2. Analysis of eigenvalues

Now that the evolution equations for constraint C_E and C_B become

$$\partial_t C_E = \alpha_1 (\Delta \lambda_E), \qquad \partial_t C_B = \alpha_2 (\Delta \lambda_B) \tag{2.16}$$

where $\Delta = \partial_i \partial^i$. We take the Fourier integrals for constraints Cs [(2.16)] and λs [(2.12), (2.13)], in the form:

$$\lambda_i = \int \hat{\lambda} e^{ik\cdot x} d^3 k, \qquad C_i = \int \hat{C} e^{ik\cdot x} d^3 k, \tag{2.17}$$

then we get

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \\ \hat{\lambda}_E \\ \hat{\lambda}_B \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha_1 k & 0 \\ 0 & 0 & 0 & -\alpha_2 k \\ \alpha_1 & 0 & -\beta_1 & 0 \\ 0 & \alpha_2 & 0 & -\beta_2 \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \\ \hat{\lambda}_E \\ \hat{\lambda}_B \end{pmatrix}, \tag{2.18}$$

where $k = k_i k^i$. We find the matrix is constant. Note that this is exact expression. Since the eigenvalues are written as $(-\beta_1 \pm \sqrt{\beta_1^2 - 4\alpha_1 k})/2$ and $(-\beta_2 \pm \sqrt{\beta_2^2 - 4\alpha_2 k})/2$, the negative eigenvalue requirement becomes $\alpha_1, \alpha_2 \neq 0$ and $\beta_1, \beta_2 > 0$.

3. Numerical demonstration

We present numerical demonstration of the above λ-system in the Maxwell system. We prepare a code which produces electromagnetic propagation in xy-plane, and monitor the violation of the constraint equation during time integration. More detail, we prepare the initial data with a Gaussian packet at the origin,

$$E^i(x, y, z) = (-Aye^{-B(x^2+y^2)}, Axe^{-B(x^2+y^2)}, 0), \tag{2.19}$$
$$B^i(x, y, z) = (0, 0, 0), \tag{2.20}$$

where A and B are constants, and let it propagate freely, under the periodic boundary condition.

The code itself is quite stable for this problem. In Fig.1, we plot L2 norm of the error (C_E over the whole grids) as a function of time. The solid line (constant line) in Fig.1 (a) is of the original Maxwell equation. If we introduce λs, then we see the error will be reduced by a particular choice of α and β. Fig.1 (a) is for changing α with $\beta = 2.0$, while Fig.1 (b) is for changing β with $\alpha = 0.5$. Here, we simply use $\alpha := \alpha_1 = \alpha_2$ and $\beta := \beta_1 = \beta_2$. We see better performance for $\beta > 0$ [Fig.1 (b)], which is the case of negative eigenvalues of the constraint propagation equation. We also see the system will diverge for large α [Fig.1 (b)], which may be explained by that the additional system's correction goes beyond perturbative regime.

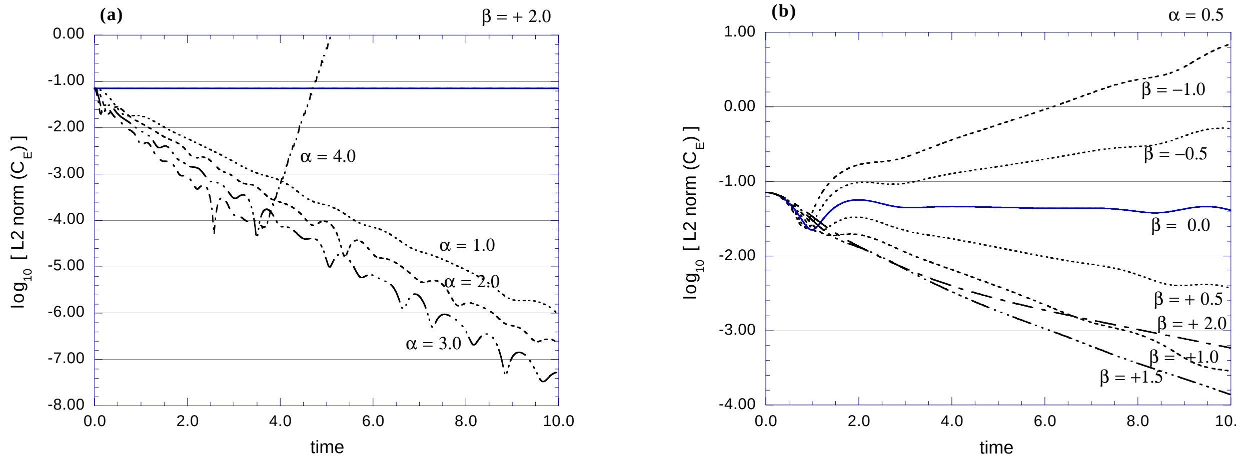


FIG. 1. Demonstrations of the λ-system in the Maxwell equation. Fig.(a) is constraint violation (L2 norm of C_E) versus time with constant $\beta(=2.0)$ but changing α. Here $\alpha=0$ means no λ-system. Fig.(b) is the same plot with constant $\alpha(=0.5)$ but changing β. We see better performance for $\beta>0$, which is the case of negative eigenvalues of the constraint propagation equation. The constants in (2.20) were chosen as $A=200$ and $B=1$.

C. Example 2: Einstein equations (Ashtekar equations)

The second demonstration is of the vacuum Einstein equations in the Ashtekar's connection formalism [5].

Before going through the λ-system, we here briefly outline the equations. The fundamental Ashtekar's new variables are the densitized inverse triad, $\tilde{E}^i_a$, and SO(3,C) self-dual connection, $\mathcal{A}^a_i$, where the indices $i,j,\cdots$ indicates the 3-spacetime, and $a,b,\cdots$ is for SO(3) space. The total four-dimensional spacetime is described together with the gauge variables $\underset{\sim}{N}, N^i, \mathcal{A}^a_0$, which we call the densitized lapse function, shift vector and the triad lapse function. Since the Hilbert action takes the form

$$S=\int \mathrm{d}^4x[(\partial_t\mathcal{A}^a_i)\tilde{E}^i_a+\underset{\sim}{N}\mathcal{C}_H+N^i\mathcal{C}_{Mi}+\mathcal{A}^a_0\mathcal{C}_{Ga}], \tag{2.21}$$

the system has three constraint equations, $\mathcal{C}_H\approx\mathcal{C}_{Mi}\approx\mathcal{C}_{Ga}\approx 0$, which are called the Hamiltonian, momentum, and Gauss constraint equation, respectively. They are written as

$$\mathcal{C}_H := (i/2)\epsilon^{ab}{}_c\,\tilde{E}^i_a\tilde{E}^j_bF^c_{ij}, \tag{2.22}$$

$$\mathcal{C}_{Mi} := -F^a_{ij}\tilde{E}^j_a, \tag{2.23}$$

$$\mathcal{C}_{Ga} := \mathcal{D}_i\tilde{E}^i_a, \tag{2.24}$$

where $F^a_{\mu\nu}:=2\partial_{[\mu}\mathcal{A}^a_{\nu]}-i\epsilon^a{}_{bc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu$ is the curvature 2-form and $\mathcal{D}_i\tilde{E}^j_a:=\partial_i\tilde{E}^j_a-i\epsilon_{ab}{}^c\,\mathcal{A}^b_i\tilde{E}^j_c$. The original dynamical equation for $(\tilde{E}^i_a,\mathcal{A}^a_i)$ constitutes a weakly hyperbolic form,

$$\partial_t\tilde{E}^i_a=-i\mathcal{D}_j(\epsilon^{cb}{}_a\,\underset{\sim}{N}\tilde{E}^j_c\tilde{E}^i_b)+2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a)+i\mathcal{A}^b_0\epsilon_{ab}{}^c\,\tilde{E}^i_c, \tag{2.25}$$

$$\partial_t\mathcal{A}^a_i=-i\epsilon^{ab}{}_c\,\underset{\sim}{N}\tilde{E}^j_bF^c_{ij}+N^jF^a_{ji}+\mathcal{D}_i\mathcal{A}^a_0 \tag{2.26}$$

where $\mathcal{D}_jX^{ji}_a:=\partial_jX^{ji}_a-i\epsilon_{ab}{}^c\mathcal{A}^b_jX^{ji}_c$, for $X^{ij}_a+X^{ji}_a=0$. It is also possible to express a set of (2.25) and (2.26) to reveal symmetric hyperbolicity [6,7]. For more detail definitions and our notation, please see Appendix A of our Paper I [1].

1. λ-system for controlling constraint violations

We here only consider the λ-system which controls the violations from the constrained surface. In [4], we have also discussed an advanced version of the λ-system which controls the violations of the reality condition.

We introduce new variables $(\lambda, \lambda_i, \lambda_a)$, as they obey the dissipative evolution equations,

$$\partial_t \lambda = \alpha_1 \, \mathcal{C}_H - \beta_1 \, \lambda, \tag{2.27}$$

$$\partial_t \lambda_i = \alpha_2 \, \tilde{\mathcal{C}}_{Mi} - \beta_2 \, \lambda_i, \tag{2.28}$$

$$\partial_t \lambda_a = \alpha_3 \, \mathcal{C}_{Ga} - \beta_3 \, \lambda_a, \tag{2.29}$$

where $\alpha_i \neq 0$ (allowed to be complex numbers) and $\beta_i > 0$ (real numbers) are constants.

If we take $y_\alpha := (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$ as a set of dynamical variables, then the principal part of (2.27)-(2.29) can be written as

$$\partial_t \lambda \cong -i\alpha_1 \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d (\partial_l \mathcal{A}^b_j), \tag{2.30}$$

$$\partial_t \lambda_i \cong \alpha_2 [-e\delta^l_i \tilde{E}^j_b + e\delta^j_i \tilde{E}^l_b](\partial_l \mathcal{A}^b_j), \tag{2.31}$$

$$\partial_t \lambda_a \cong \alpha_3 \partial_l \tilde{E}^l_a. \tag{2.32}$$

The characteristic matrix of the system u_α does not form a Hermitian matrix. However, if we modify the right-hand-side of the evolution equation of $(\tilde{E}^i_a, \mathcal{A}^a_i)$, then the set becomes a symmetric hyperbolic system. This is done by adding $\bar{\alpha}_3 \gamma^{il}(\partial_l \lambda_a)$ to the equation of $\partial_t \tilde{E}^i_a$, and by adding $i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d (\partial_l \lambda) + \bar{\alpha}_2(-e\gamma^{lm} \tilde{E}^a_i + e\delta^m_i \tilde{E}^{la})(\partial_l \lambda_m)$ to the equation of $\partial_t \mathcal{A}^a_i$. The final principal part, then, is written as

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} \mathcal{M}^l{}_a{}^{bi}{}_m & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il} \delta_a{}^b \\ 0 & \mathcal{N}^{la}{}_{ib}{}^m & i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e(\delta^m_i \tilde{E}^{la} - \gamma^{lm} \tilde{E}^a_i) & 0 \\ 0 & -i\alpha_1 \epsilon_b{}^{cd} \tilde{E}^m_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e(\delta^m_i \tilde{E}^l_b - \delta^l_i \tilde{E}^m_b) & 0 & 0 & 0 \\ \alpha_3 \delta_a{}^b \delta^l{}_m & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^m_b \\ \mathcal{A}^b_m \\ \lambda \\ \lambda_m \\ \lambda_b \end{pmatrix}. \tag{2.33}$$

where

$$\mathcal{M}^{labij} = i\epsilon^{abc} \underset{\sim}{N} \tilde{E}^l_c \gamma^{ij} + N^l \gamma^{ij} \delta^{ab}, \tag{2.34}$$

$$\begin{aligned} \mathcal{N}^{labij} &= i\underset{\sim}{N}(\epsilon^{abc} \tilde{E}^j_c \gamma^{li} - \epsilon^{abc} \tilde{E}^l_c \gamma^{ji} - e^{-2} \tilde{E}^{ia} \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d - e^{-2} \epsilon^{acd} \tilde{E}^i_d \tilde{E}^l_c \tilde{E}^{jb} \\ &\quad + e^{-2} \epsilon^{acd} \tilde{E}^i_d \tilde{E}^j_c \tilde{E}^{lb}) + N^l \delta^{ab} \gamma^{ij}, \end{aligned} \tag{2.35}$$

Clearly, the solution $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a) = (\tilde{E}^i_a, \mathcal{A}^a_i, 0, 0, 0)$ represents the original solution of the Ashtekar system.

2. Analysis of eigenvalues

The propagation equation of the constraints $(\mathcal{C}_H, \tilde{\mathcal{C}}_{Mi}, \mathcal{C}_{Ga})$ and their indicators $(\lambda, \lambda_i, \lambda_a)$ are written as, after linearized and taken these Fourier transformation (2.17),

$$\partial_t \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\tilde{\mathcal{C}}}_{Mi} \\ \hat{\mathcal{C}}_{Ga} \\ \hat{\lambda} \\ \hat{\lambda}_i \\ \hat{\lambda}_a \end{pmatrix} = \begin{pmatrix} 0 & -ik_j & 0 & -2\bar{\alpha}_1 k_m k^m & 0 & 0 \\ -ik_i & k_m \epsilon^m{}_i{}^j & 0 & 0 & -\bar{\alpha}_2(k_i k^j + k_m k^m \delta^{ji}) & 0 \\ 0 & -2\delta^b_a & \epsilon^{mb}{}_a k_m & 2i\bar{\alpha}_1 k_a & \bar{\alpha}_2 \epsilon^{amj} k_m & -\bar{\alpha}_3 k_m k^m \delta^b_a \\ \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 \delta^j_i & 0 & 0 & -\beta_2 \delta^j_i & 0 \\ 0 & 0 & \alpha_3 \delta^b_a & 0 & 0 & -\beta_3 \delta^b_a \end{pmatrix} \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\tilde{\mathcal{C}}}_{Mj} \\ \hat{\mathcal{C}}_{Gb} \\ \hat{\lambda} \\ \hat{\lambda}_j \\ \hat{\lambda}_b \end{pmatrix}, \tag{2.36}$$

In order to link the discussion with our later demonstration in the plane symmetric spacetime, we here consider only the Fourier component of $k_i = (1, 0, 0)$ for simplicity. The eigenvalues, $E_i \quad (i = 1, \cdots, 14)$, of the characteristic matrix of (2.36) can be written explicitly as

$$
\begin{aligned}
(E_1, \cdots, E_{10}) = & -(1/2)\beta_3 \pm (1/2)\sqrt{\beta_3^2 - 4|\alpha_3|^2}, \\
& -(1/2)(i+\beta_3) \pm (1/2)\sqrt{-1 - 4|\alpha_3|^2 - 2i\beta_3 + \beta_3^2}, \\
& -(1/2)(-i+\beta_3) \pm (1/2)\sqrt{-1 - 4|\alpha_3|^2 - 2i\beta_3 + \beta_3^2}, \\
& -(1/2)(i+\beta_2) \pm (1/2)\sqrt{-1 - 4|\alpha_2|^2 - 2i\beta_2 + \beta_2^2}, \\
& -(1/2)(-i+\beta_2) \pm (1/2)\sqrt{-1 - 4|\alpha_2|^2 - 2i\beta_2 + \beta_2^2}
\end{aligned}
$$

and as solutions $(E_{11}, \cdots, E_{14})$ of the quartic equation

$$
x^4 + (\beta_2 + \beta_1)x^3 + (2|\alpha_1|^2 + 2|\alpha_2|^2 + 1 + \beta_1\beta_2)x^2 + (2|\alpha_2|^2\beta_1 + \beta_2 + \beta_1 + 2|\alpha_1|^2\beta_2)x + (\beta_1\beta_2 + 4|\alpha_1|^2|\alpha_2|^2) = 0, \tag{2.37}
$$

where $|\alpha_i|^2 = \alpha_i\bar{\alpha}_i$. We omit the explicit expressions of $E_{11}, \cdots, E_{14}$ in order to save the space.

The conditions for $\alpha_\rho, \quad \beta_\rho, \quad (\rho = 1, 2, 3)$ to make $\Re e(E_i) < 0$ are suggested as

$$
\alpha_\rho \neq 0 \qquad \text{and} \qquad \beta_\rho > 0. \tag{2.38}
$$

This is true (necessary and sufficient) for $E_1, \cdots, E_{10}$, and also plausible for $E_{11}, \cdots, E_{14}$ as far as our numerical evaluation tells (see Fig.2).

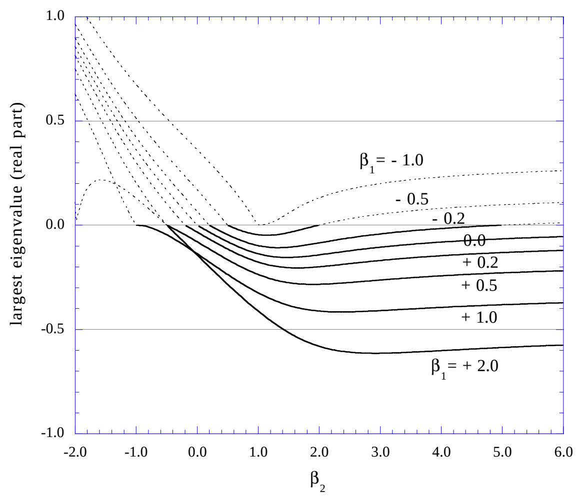


FIG. 2. Example of eigenvalues of the system (2.36). We plot the eigenvalue which has the maximum real part between four of them, for the case of fixing $\alpha_1 = \alpha_2 = 1$ and changing β_1 and β_2. We see our desired condition, "all negative real eigenvalues", is available when the combinations produce the solid lines. That is, when both β take the large positive values.

3. Numerical demonstration

In this subsection, we demonstrate that the λ-system for the Ashtekar equation actually works.

The model we present here is gravitational wave propagation in a planar spacetime, using a full numerical simulation using Ashtekar's variables. We prepare two +-mode strong pulse waves initially by solving the Hamiltonian constraint of Arnowitt-Deser-Misner's 3+1 formulation, using York-O'Murchadha's conformal approach. Then we transform the initial Cauchy data (3-metric and extrinsic curvature) into Ashtekar's connection variables, $(\tilde{E}^i_a, \mathcal{A}^a_i)$, and evolve them using these dynamical equations (either its original form or its symmetric hyperbolically reformulated form). The details of the numerical method are described in the Paper I [1].

In order to show an expected "stabilization behavior" clearly, we artificially add an error in the middle of time evolution. More specifically, we set out initial guess 3-metric as

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 + 0.3(e^{-(x-2.5)^2} + e^{-(x+2.5)^2}) & 0 \\ & & 1 - 0.3(e^{-(x-2.5)^2} + e^{-(x+2.5)^2}) \end{pmatrix}, \tag{2.39}$$

in the periodically bounded region $x = [-5, +5]$, and added an artificial inconsistent rescaling once at time $t = 6$ for $\mathcal{A}^2_y$ component as $\mathcal{A}^2_y = \mathcal{A}^2_y(1 + \text{error})$.

Fig.3 (a) shows how the violation of the Hamiltonian constraint equation, $\mathcal{C}_H$, become worse depending the term error. The oscillation of the L2 norm $\mathcal{C}_H$ in the figure due to the pulse waves collide periodically in the numerical region. We, then, fix the error term as 20% spike, and try to evolve the same data in different equations of motion, i.e., the original Ashtekar's equation [solid line in Fig.3 (b)], strongly hyperbolic version of Ashtekar's equation (dotted line) and the above λ-system equation (other lines) with different βs but the same α. As we expected, all the λ-system cases result in reducing an excited Hamiltonian constraint errors.

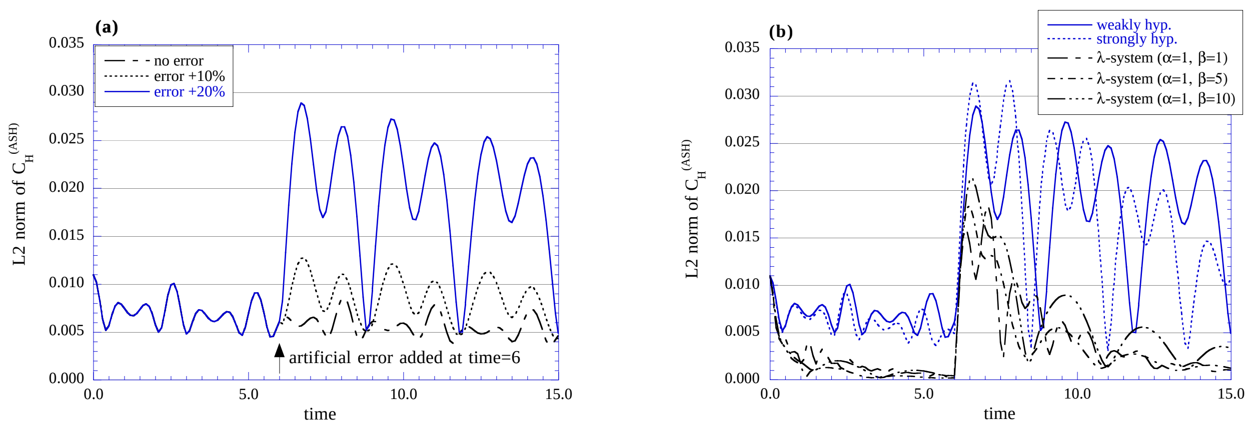


FIG. 3. Demonstration of the λ-system in the Ashtekar equation. We plot the violation of the constraint (L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$) for the cases of plane wave propagation under the periodic boundary. To see the effect more clearly, we added artificial error at $t = 6$. Fig. (a) shows how the system go worse depending on the amplitude of artificial error. Error was like a kick of $\mathcal{A}^2_y = \mathcal{A}^2_y(1 + \text{ error})$. All the lines are of the evolution by Ashtekar's original equation (no λ-system). Fig. (b) shows the effect of λ-system. All the lines are of 20% error amplitude, but shows the differences of evolution equations. The solid line is of Ashtekar's original equation (the same as in Fig.(a)), the dotted line is of strongly hyperbolic Ashtekar's equation. Other lines are of λ-system, which produce better performance than that of the strongly hyperbolic system.

D. Remarks for the λ-system

In the previous subsections, we showed that λ-system works as we expected. The system evolves into a constrained surface asymptotically even if we added an error artificially. However, the λ-system can not be introduced generally, because (i) the construction of the λ-system requires the original dynamical equation in the symmetric hyperbolic form, which is quite restrictive for the Einstein equation, (ii) the system requires additional many variables and we also need to evaluate all constraint equation at every time steps, which are numerically hard tasks. Moreover, the λ-system may efficiently work only for the constraints which are written in spatial differential terms.

We, next, propose an alternative system which also enable us to control the violation of constraint equations, but robust for these points.

III. ASYMPTOTICALLY CONSTRAINED SYSTEM 2: ADJUSTED SYSTEM

We present here another approach for obtaining stable evolutions, which we named "adjusted-system". The essential procedures are to add constraint terms in the right-hand-side of the dynamical equations with multipliers, and to choose the multipliers so as they decrease the violation of constraint equation. This system has several advantages than the previous λ-system.

A. "Adjusted system"

The actual procedures for constructing an adjusted system are followings.

(1) Prepare a set of evolution equations for dynamical variables and these first class constraint which describe the problem. It is not required that the system is in the first order form nor hyperbolic form. However here we start from the same form with (2.1) and (2.2). We repeat them as

$$\partial_t u^\gamma = A^{i\gamma}{}_\delta \partial_i u^\delta + B^\gamma, \tag{3.1}$$
$$\partial_t C^\rho = D^{i\rho}{}_\sigma \partial_i C^\sigma + E^\rho{}_\sigma C^\sigma, \tag{3.2}$$

where $A(u(x^i))$ is not required to form a symmetric or Hermitian matrix.

(2) Add the constraint terms, C^ρ, (and/or its derivative terms) to the dynamical equation (3.1) with multipliers κ,

$$\partial_t u^\gamma = A^{i\gamma}{}_\delta \partial_i u^\delta + B^\gamma + \kappa^\gamma_\rho C^\rho + \kappa^{\gamma i}_\rho \partial_i C^\rho. \tag{3.3}$$

We call the added terms, $\kappa^\gamma_\rho C^\rho$ and/or $\kappa^{\gamma i}_\rho \partial_i C^\rho$, "adjusted terms", and let κ^γ_ρ and $\kappa^{\gamma i}_\rho$ be unknown for a while. Because of these adjusted terms, the original constraint propagation equations, (3.2), are also adjusted as

$$\partial_t C^\rho = D^{i\rho}{}_\sigma \partial_i C^\sigma + E^\rho{}_\sigma C^\sigma + F^{ij\rho}{}_\sigma \partial_i \partial_j C^\sigma + G^{i\rho}{}_\sigma \partial_i C^\sigma + H^\rho{}_\sigma C^\sigma. \tag{3.4}$$

The last three terms are due to the adjusted terms.

(3) Specify the multipliers κ. For this process, we propose two guidelines.

(a) The first one is to obtain *negative* real-part of the eigenvalues of the characteristic part of (3.4). This is from the same principle in λ-system when we specified α and β, in order to guarantee the system approaches asymptotically constrained surface. By taking the Fourier transformation (2.7), we can reduce (3.4) in a homogeneous form,

$$\partial_t \hat{C}^\rho = (ik_i D^{i\rho}{}_\sigma + E^\rho{}_\sigma - k_i k_j F^{ij\rho}{}_\sigma + ik_i G^{i\rho}{}_\sigma + H^\rho{}_\sigma)\hat{C}^\sigma. \tag{3.5}$$

Provided that we obtain κ which will produce all the negative real-part eigenvalues of the principal part of RHS of (3.5), the Fourier component $\hat{C}$ decays to zero in time evolution, and the original constraint term C also. Practically, we may only need to evaluate the principal order of the RHS of (3.5), i.e. the eigenvalues of $(ik^{(0)}D + {}^{(0)}E - k^{(0)}F + ik_i{}^{(0)}G + {}^{(0)}H)$.

(b) Alternative guideline is to obtain *non-zero* eigenvalues of the characteristic part of (3.4). More precisely, this case is supposed to have *pure imaginary* eigenvalues. In such a case, the constraint propagation equations (e.g. $\partial_t \hat{C} = \pm ik\hat{C}$) behave like the normal wave equations in its original component (e.g. $\partial_t C = \pm \partial_x C$), and its stability property can be discussed by von Neumann stability analysis. As is well known, the stability property is depend on the choice of numerical integration scheme, but it is also certain that we can control (or decrease) the amplitude of the constraint terms.

The advantage of this adjusted system is that we do not need additional variables in the fundamental set, while the above first guideline (3a) is the same mechanism which was applied in the λ-system. We note that the *non-zero* eigenvalue feature was conjectured in [13] in order to show the advantage of conformally scaled ADM system, but their eigenvalues were of dynamical equations and not of constraint propagation equations.

We remark that adding constraint terms in the dynamical equations is not a new idea. For example, Detweiler [8] applied this procedure to ADM equation and used the finiteness of the norm to obtain a new system. Or this is also one of the standard procedure to construct a symmetric hyperbolic system (e.g. [6]). We think, however, our above guidelines are pointing the essential mechanism for our purpose, to constructing a stable dynamical system.

In the following subsections and in the Appendix, we will demonstrate that this adjusted system procedure actually works for Maxwell system and Ashtekar system of the Einstein equation, in which above two guidelines are applied respectively.

B. Example 1: Maxwell equations

1. adjusted system

We here again consider Maxwell equation (2.9)-(2.11). We start from adjusted dynamical equations

$$\partial_t E_i = c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + p^j{}_i(\partial_j C_E) + Q_i C_B + q^j{}_i(\partial_j C_B), \tag{3.6}$$

$$\partial_t B_i = -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + r^j{}_i(\partial_j C_E) + S_i C_B + s^j{}_i(\partial_j C_B), \tag{3.7}$$

where P, Q, R, S, p, q, r and s are multipliers. These dynamical equations adjust the constraint propagation equations as

$$\begin{aligned}\partial_t C_E = &+(\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B)\\ &+(\partial_i p^j{}_i)(\partial_j C_E) + p^j{}_i(\partial_i\partial_j C_E) + (\partial_i q^j{}_i)(\partial_j C_B) + q^j{}_i(\partial_i\partial_j C_B),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\partial_t C_B = &+(\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B)\\ &+(\partial_i r^j{}_i)(\partial_j C_E) + r^j{}_i(\partial_i\partial_j C_E) + (\partial_i s^j{}_i)(\partial_j C_B) + s^j{}_i(\partial_i\partial_j C_B).\end{aligned} \tag{3.9}$$

This will be expressed using Fourier components by

$$\partial_t \begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix} = \begin{pmatrix}\partial_i P^i + iP^i k_i + ik_j(\partial_i p^j{}_i) - k_i k_j p^j{}_i & \partial_i Q^i + iQ^i k_i + ik_j(\partial_i q^j{}_i) - k_i k_j q^j{}_i\\ \partial_i R^i + iR^i k_i + ik_j(\partial_i r^j{}_i) - k_i k_j r^j{}_i & \partial_i S^i + iS^i k_i + ik_j(\partial_i s^j{}_i) - k_i k_j s^j{}_i\end{pmatrix} \partial_l \begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix} =: T\begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix}. \tag{3.10}$$

Since we suppose the multipliers are constants or functions of E and B, we can truncate the principal matrix as

$${}^{(0)}T = \begin{pmatrix} iP^i k_i - k_i k_j p^j{}_i & iQ^i k_i - k_i k_j q^j{}_i\\ iR^i k_i - k_i k_j r^j{}_i & iS^i k_i - k_i k_j s^j{}_i\end{pmatrix}, \tag{3.11}$$

and its eigenvalues are obtained as

$$\Lambda^{\pm} = \frac{p + s \pm \sqrt{p^2 + 4\,q\,r - 2\,p\,s + s^2}}{2}, \tag{3.12}$$

where $p := iP^i k_i - k_i k_j p^j{}_i,\ q := iQ^i k_i - k_i k_j q^j{}_i,\ r := iR^i k_i - k_i k_j r^j{}_i,\ s := iS^i k_i - k_i k_j s^j{}_i$. If we fix $q = r = 0$, then $\Lambda^{\pm} = p, s$. Further if we assume $p^j{}_i, s^j{}_i > 0$, and set else zero, then $\Lambda^{\pm} < 0$, that is we can get the all eigen values which have negative real part. (Conversely, if we choose $p^j{}_i, s^j{}_i < 0$, then $\Lambda^{\pm} < 0$.) That is, our guideline (a) is obtained.

2. Numerical Demonstration

We applied the above adjusted system to the same wave propagation problem as in §II B 3. For simplicity, we fix $\kappa = p^j{}_i = s^j{}_i$ and set other multipliers equal to zero. In Fig.4, we show the L2 norm of constraint violation as a function of time, with various κ. As was expected, we see better performance for $\kappa > 0$ (of the system with negative real part of constraint propagation equation), while diverging behavior for $\kappa < 0$ (of the system with positive real part of constraint propagation equation).

C. Example 2: Einstein equations (Ashtekar equations)

1. Adjusted system for controlling constraint violations

We here only consider the adjusted system which controls the violations from the constrained surface. In Appendix, we present an advanced system which controls the violation of reality condition together with numerical demonstration.

Even if we restrict ourselves to adjusted equations of motion for $(\tilde{E}^i_a, \mathcal{A}^a_i)$ with constraint terms (no their derivatives), generally, we could adjust them as

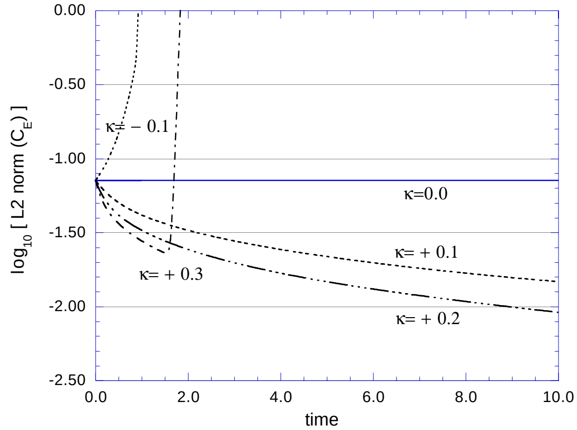


FIG. 4. Demonstrations of the adjusted system in the Maxwell equation. We perform the same experiments with §II B 3 [Fig.1]. Constraint violation (L2 norm of C_E) versus time are plotted for various $\kappa(= p^j{}_i = s^j{}_i)$. We see $\kappa > 0$ has better performance, which is the case of negative real part eigenvalues of its constraint propagation equation, while too large positive κ turns into diverging again.

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}, \tag{3.13}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i + Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}, \tag{3.14}$$

where $X^i_a, Y^{ij}_a, Z^{ab}_i, P^{ib}_a, Q^a_i$ and R^{aj}_i are multipliers. However, in order to simplify the discussion, we restrict multipliers so as they re-produce the symmetric hyperbolic equation of motion [6,7], i.e.,

$$X = Y = Z = 0, \qquad P^{ia}_b = \kappa_1 (iN^i \delta^a_b), \quad Q^a_i = \kappa_2 (e^{-2} \underset{\sim}{N} \tilde{E}^a_i), \quad R^{aj}{}_i = \kappa_3 (-ie^{-2} \underset{\sim}{N} \epsilon^{ac}{}_d \tilde{E}^d_i \tilde{E}^j_c). \tag{3.15}$$

Here $\kappa_1 = \kappa_2 = \kappa_3 = 1$ is the case of symmetric hyperbolic equation for $(\tilde{E}^i_a, \mathcal{A}^a_i)$, while $\kappa_1 = \kappa_2 = \kappa_3 = 0$ is the (Ashtekar's original) weakly hyperbolic equation, and other choices of κs let the equation satisfy the level of strongly hyperbolic form.

With these adjusted terms, the constraint propagation equations become

$$\partial_t {}^{(0)}\mathcal{C}_H = (\partial_j {}^{(0)}\mathcal{C}_{Mj}) - 2\kappa_3 (\partial_k {}^{(0)}\mathcal{C}_{Mk}) = (1 - 2\kappa_3)(\partial_j {}^{(0)}\mathcal{C}_{Mj}), \tag{3.16}$$

$$\partial_t {}^{(0)}\mathcal{C}_{Mi} = (\partial_i \mathcal{C}_H)(\partial_i {}^{(0)}\mathcal{C}_H) + i\kappa_3 \epsilon^{aj}{}_i (\partial_a {}^{(0)}\mathcal{C}_{Mj}) = (1 - 2\kappa_2)(\partial_i {}^{(0)}\mathcal{C}_H) + i\kappa_3 \epsilon^{mj}{}_i (\partial_m {}^{(0)}\mathcal{C}_{Mj}), \tag{3.17}$$

$$\partial_t {}^{(0)}\mathcal{C}_{Ga} = i\kappa_1 \epsilon_a{}^{bi} (\partial_i {}^{(0)}\mathcal{C}_{Gb}) - 2\kappa_3 {}^{(0)}\mathcal{C}_{Ma} = -2\kappa_3 {}^{(0)}\mathcal{C}_{Ma} + i\kappa_1 \epsilon_a{}^{bm} (\partial_m {}^{(0)}\mathcal{C}_{Gb}). \tag{3.18}$$

The eigenvalues of these Fourier-transformed characteristic matrix are obtained as

$$(0, \quad \pm i\kappa_1 \sqrt{k}, \quad \pm i\kappa_3 \sqrt{k}, \quad \pm i(2\kappa_2 - 1)(2\kappa_3 - 1)\sqrt{k}) \tag{3.19}$$

where $k = k_i k^i$. For example,

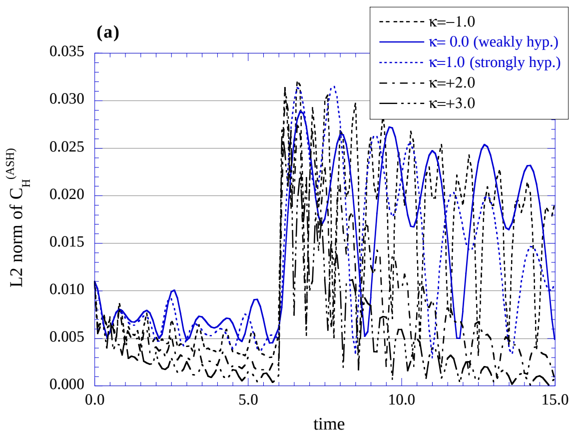
$$(0 \quad (5 \text{ multiplicity}), \quad \pm i\sqrt{k}) \qquad \text{for } \kappa_1 = \kappa_2 = \kappa_3 = 0 \quad : \text{original system} \tag{3.20}$$

$$(0, \quad \pm i\sqrt{k} \quad (3 \text{ multiplicity each})) \qquad \text{for } \kappa_1 = \kappa_2 = \kappa_3 = 1 \quad : \text{symmetric hyperbolic system.} \tag{3.21}$$

That is, our guideline (b) is obtained.

The above way of adjustment, (3.13)-(3.15), will not produce negative real-part of eigenvalues, so that our guideline (a) can not be applied here. If we adjusted the dynamical equation using the spatial derivatives of constraint terms, then it is possible to get all negative eigenvalues like in the Maxwell system (though it is complicated). However, since we found that this adjustment, (3.13)-(3.15), gives us an example for controlling the violation of constraint equations for our purpose, we only show this simpler version here.

As a demonstration, we here again use the same model as in §II C 3, that is the gravitational wave propagation in the plane symmetric spacetime, with an artificial error in the middle of time evolution. We examine how the adjusted multipliers contribute the system's stabilities. In Fig.5, we show the results of this experiments. We plot the violation of the constraint equations both $\mathcal{C}_H$ and $\mathcal{C}_{Mx}$. An artificial error term was added at $t = 6$, as a kick of $\mathcal{A}^2_y = \mathcal{A}^2_y(1 + \text{ error})$, where error amplitude is $+20\%$ as before. The solid line is the case of $\kappa = 0$, that is the case of "no adjusted" original Ashtekar equation (weakly hyperbolic system). The dotted line is of $\kappa = 1$, and equivalent with strongly hyperbolic system. We see other lines ($\kappa = 1.5, 2.0$) shows better performance than the symmetric hyperbolic case.



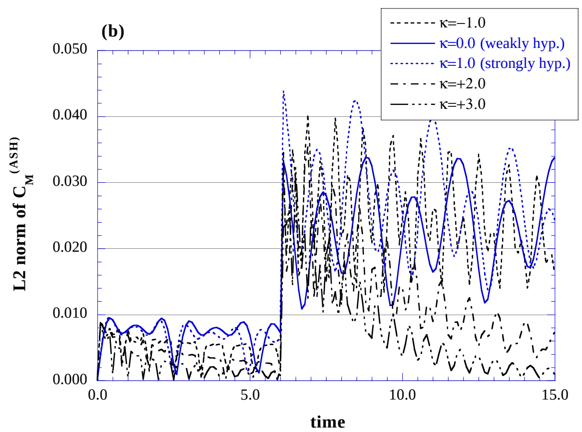


FIG. 5. Demonstration of the adjusted system in the Ashtekar equation. We plot the violation of the constraint for the same model with Fig.3(b). An artificial error term was added at $t = 6$, as a kick of $\mathcal{A}^2_y = \mathcal{A}^2_y(1 + \text{ error})$, where error is $+20\%$ as before. Fig. (a) and (b) are L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$, and momentum constraint equation, $\mathcal{C}_{Mx}$, respectively. The solid line is the case of $\kappa = 0$, that is the case of "no adjusted" original Ashtekar equation (weakly hyperbolic system). The dotted line is of $\kappa = 1$, and equivalent with strongly hyperbolic system. We see other lines ($\kappa = 1.5, 2.0$) shows better performance than the symmetric hyperbolic case.

IV. DISCUSSION

With the purpose of searching for an evolution system of the Einstein equation which is robust against perturbative errors for the free evolution of the initial data, we studied two "asymptotically constrained" systems.

We, first, examined the previously proposed "λ-system", which introduces artificial flows to constrained surfaces based on the symmetric hyperbolic formulation. We showed that this system works as expected for the wave propagation problem in the Maxwell system and in the Ashtekar's system of general relativity. However, the λ-system can not be applied for general dynamical systems in general relativity, since the system requires the base system to be symmetric hyperbolic form.

Alternatively, we proposed a new mechanism to control the stability, which we named "adjusted system". This is simply obtained by adding constraint terms in the dynamical equations and adjusting its multipliers. We proposed two guidelines for specifying multipliers which reduce the numerical errors; that is, non-positiveness or non-zero (pure-imaginary) of the eigenvalues of the adjusted constraint propagation equations. This adjusted system was also tested in the Maxwell system and in the Ashtekar's system.

As we denoted earlier, the idea of adding constraint terms is not new. However, we think that our guidelines for controlling the decay of constraint equations are matched for our purposes which were not pointed out before. Up to our numerical experiments, our guidelines give us clear judgement whether the constraint decays or not (stable system or not) for perturbative errors, though we also think that this is not the final explanation for all the cases.

This feature may be explained or proven in different ways, such as finiteness of the norm (of evolution equations or of constraint propagation equations), or by another mechanism in future.

Our by-product conclusion is that the symmetric hyperbolic equation is not always the best one for controlling stable evolution. As we shown in the wave propagation model in the adjusted Ashtekar's equation, our eigenvalue guidelines affects more than the system's hyperbolicity. (We found similar conclusion in [14].) We think this result opens a new direction to numerical relativists for future treatment of the Einstein equation.

We are now applying our idea to the standard ADM and conformally scaled ADM system to explain these differences. Results will be reported elsewhere [15].

ACKNOWLEDGEMENTS

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APPENDIX: CONTROLLING REALITY CONDITION BY ADJUSTED SYSTEM

We demonstrate here that our adjusted system in the Ashtekar formulation also works for controlling reality conditions. As a model problem, we concern the degenerate point passing problem which were considered by ourselves in [16]. In §A 1, we review this background briefly, and in §A 2 we show our numerical demonstrations.

1. Degenerate point passing problem

In [16], the authors had examined the possibility of dynamical passing of the degenerate point in the spacetime. There the authors found that we are able to pass (i.e. continue time evolutions) if we could foliate the time-constant hypersurface into complex plane assuming that such a degenerate point exists on the real plane. Such foliations are available within Ashtekar's original formulation, since the fundamental variables are complex quantities. The trick is to violate the reality condition locally, only at the vicinity of a degenerate point.

More detail is the following. As a model, we construct a metric, ${}^{(4)}g$, which possesses a degenerate point ($\det {}^{(3)}g = 0$) at the origin $t = x = 0$ in Minkowskii background metric:

$$\begin{aligned} ds^2 = &-[1-(2tx\exp(-t^2-x^2))^2]dt^2 + 4tx\exp(-t^2-x^2)[1-(1-2x^2)\exp(-t^2-x^2)]dtdx \\ &+[1-(1-2x^2)\exp(-t^2-x^2)]^2dx^2 + dy^2 + dz^2. \end{aligned} \tag{A1}$$

We consider the time evolution, which initial data is described by a particular time slice $t < 0$ of (A1), and its time-constant hypersurfaces are foliated by the gauge condition,

$$N = 1, \ (\underset{\sim}{N} = e^{-1}), \tag{A2}$$

$$N_x = 2tx\exp(-t^2-x^2)[1-(1-2x^2)\exp(-t^2-x^2)] + iat\exp(-b(t^2+x^2)), \tag{A3}$$

$$\mathcal{A}_0^a = 0, \tag{A4}$$

which enables to detour in complex plane. Our goal is to demonstrate the time evolution that come back to the real plane without any divergence in variables and curvatures. Such a "recovering condition" can be described by

$$\text{Foliation recovering condition} \quad \int_{t_-}^{t_+} \Im N(t,\boldsymbol{x})dt = 0, \ \int_{t_-}^{t_+} \Im N^i(t,\boldsymbol{x})dt = 0, \tag{A5}$$

$$\text{Asymptotic reality condition} \quad \Im N(t,\boldsymbol{x}) \to 0, \ \Im N^i(t,\boldsymbol{x}) \to 0, \ \Im\left[\tilde{E}^i_a \tilde{E}^{ja}/\det\tilde{E}(t,\boldsymbol{x})\right] \to 0, \tag{A6}$$

for all four limits $\boldsymbol{x} \to \boldsymbol{x}_* \pm \Delta\boldsymbol{x}$, $t \to t_* \pm \Delta t$.

Numerically, this problem turned into an eigenvalue problem, since our boundary conditions, (A5) and (A6), specify much freedom. To see the evolution satisfies the criteria or not, we introduced two measures

$$F(t_{final}) := \max_x |\Re\left(e(t = t_{final}, x) - 1\right)| \quad \text{(asymptotically flatness)} \tag{A7}$$

$$R(t_{final}) := \max_x |\Im\left(e(t = t_{final}, x)\right)| \quad \text{(asymptotically reality)} \tag{A8}$$

and searched the parameters a and b in (A3).

If we applied our adjusted system to this model, then we expect that allowed parameters range for a and b becomes more general, since the real-surface recovering feature is in the flow of the adjusted system's foliation.

2. Application of adjusted system

As was shown in the previous section, for this purpose, we have to foliate our hypersurface in complex-valued region and foliate back to the real-valued surface. That is, we can treat the reality condition, both primary and secondary, as a part of constraint equations.

For the above degenerate point passing problem, we need to control only the violation of $\Im m(\tilde{E}^i_a \tilde{E}^j_a)$. Therefore, similar to the proposal of adjusted system discussed in §II C, our adjusted dynamical equations can be written as

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb} + T^i{}_{ajk} \Im m(\tilde{E}^j_b \tilde{E}^k_b), \tag{A9}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i + Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb} + V^a{}_{ijk} \Im m(\tilde{E}^j_b \tilde{E}^k_b), \tag{A10}$$

where $X^i_a, Y^{ij}_a, Z^{ab}_i, P^{ib}_a, Q^a_i, R^{aj}_i, T^i{}_{ajk}$ and $V^a{}_{ijk}$ are adjusted multipliers.

If we set simply $X^i_a = Y^{ij}_a = Z^{ab}_i = P^{ib}_a = Q^a_i = R^{aj}_i = V^a{}_{ijk} = 0$ and $T^i{}_{ajk} = -i\kappa\delta^i_j\delta_{ak}$ where κ is real constant, then we obtain the constraint propagation equation as

$$\partial_t({}^{(0)}\Im m(\tilde{E}^i_a \tilde{E}^j_a)) = -2\kappa({}^{(0)}\Im m(\tilde{E}^i_a \tilde{E}^j_a)) + \text{other constraint terms.} \tag{A11}$$

The eigenvalue of this Fourier-transformed RHS is -2κ. That is, if we set $\kappa > 0$ (< 0) then the eigenvalue is negative (positive), while $\kappa = 0$ recovers the original non-adjusted system.

The results of numerical demonstration are shown in Fig.6. We plot L2 norm of violation of reality condition as a function of time, t (this evolution is from $t = -5$, to 5 [16]). Around the time $t = 0$ the error appears due to our "detour" slicing condition, and the original system ($\kappa = 0$) will not recover the reality surface with the choice of a and b in (A3) for this plot. However, for positive κ case, the foliation will be forced to recover the reality surface, while for negative κ case will not.

Therefore this example again supports our guidelines, i.e. negative eigenvalue of constraint propagation equation will guarantee the evolution to the constrained surface.

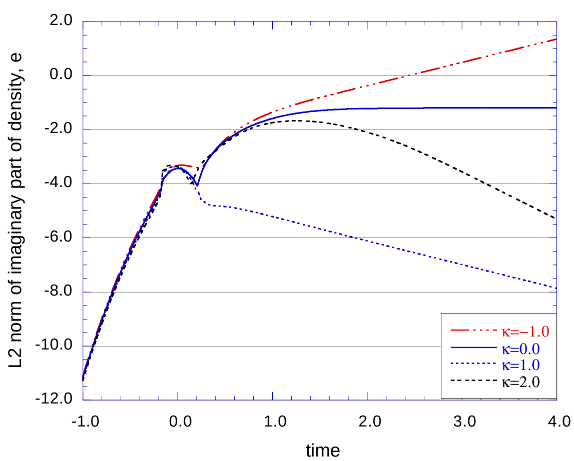

FIG. 6. Demonstration of the adjusted system to control the reality condition in the Ashtekar formulation. Reality violation (L2 norm of imaginary part of density) versus time are plotted for various adjusted coefficient $\kappa = -1, 0, 1, 2$. We see $\kappa > 0$ has better performance, which is the case of negative real part eigenvalues of its reality propagation equation, (A11).

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[1] H. Shinkai and G. Yoneda, "Hyperbolic formulations and numerical relativity: Experiments using Ashtekar's connection variables", gr-qc/0005003. [Paper I]
[2] O. Brodbeck, S. Frittelli, P. Hübner and O.A. Reula, J. Math. Phys. **40**, 909 (1999).
[3] S. Frittelli and O.A. Reula, Phys. Rev. Lett. **76**, 4667 (1996). See also J.M. Stewart, Class. Quant. Grav. **15**, 2865 (1998).
[4] H. Shinkai and G. Yoneda, Phys. Rev. **D60**, 101502 (1999).
[5] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. **D36**, 1587 (1987); *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).
[6] G. Yoneda and H. Shinkai, Phys. Rev. Lett. **82**, 263 (1999).
[7] G. Yoneda and H. Shinkai, Int. J. Mod. Phys. D. **9**, 13 (2000).
[8] S. Detweiler, Phys. Rev. **D35**, 1095 (1987).
[9] T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Suppl. **90**, 1 (1987). M. Shibata and T. Nakamura, Phys. Rev. **D52**, 5428 (1995).
[10] T. W. Baumgarte and S. L. Shapiro, Phys. Rev. **D59**, 024007 (1999). T. W. Baumgarte, S.A. Hughes, and S. L. Shapiro, *ibid.* **D60**, 087501 (1999).
[11] M. Alcubierre, *et al.* gr-qc/0003071.
[12] S. Frittelli, Phys. Rev. **D55**, 5992 (1997).
[13] M. Alcubierre, G. Allen, B. Brügmann, E. Seidel and W-M. Suen, gr-qc/9908079.
[14] S. D. Hern, PhD thesis, gr-qc/0004036.
[15] H. Shinkai and G. Yoneda, in preparation.
[16] G. Yoneda, H. Shinkai and A. Nakamichi, Phys. Rev. **D56**, 2086 (1997).